Elements of dodecahedral cosmology.

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Abstract

A general method for the determination of the harmonics of quotients of the 3-sphere is given. They can all be deduced from three objects already known from Klein. We further show explicitly how these harmonics can be organized in irreducible representations of the holonomy group: this allows for the determination of the full correlation matrix of the Cosmic Microwave Background in the spherical harmonics basis.

1 Introduction

The question of the global structure of our universe is a basic question in cosmology. Inflationary models of the universe suppose that the expansion is so great that the observable part of the universe will for ever remain negligible and global aspects firmly out of the reach of observation. However the low values of the quadrupole and octopole of the cosmological microwave background observed by WMAP [1] suggested that the universe may not be so large and a case has been made for its structure to be that of a dodecahedral quotient of the three-sphere in [2]. These low values are by themselves no proof of such a finite universe, since other explanations have been proposed, the simplest one being that these low values are compatible with cosmic variance. It is therefore important to have a more compelling argument for a small compact universe, which also allows to choose between the different proposed geometries.

The purpose of this paper is to show that a general method to describe the harmonics can be deduced from the classical results of Klein [3] on the invariant polynomials for subgroups of $SU(2)$. Furthermore the correlations between different multipoles become computable, when using the decomposition of the representation of $SU(2)$ in irreducible representations of the binary polyhedral group. The count of these representations is known from the study of Kostant [4, 5], but explicit basis for these representations are necessary and will be presented here. Their constructions owe much to the methods of Klein.

In the study of the cosmological microwave background in a given geometry, the eigenmodes of the Laplacian on this space have to be determined. In the

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case of the quotients of the three-sphere, the eigenvalues with their multiplicities have been determined by considerations on the characters of the binary polyhedral groups on the space of spherical harmonics [6]. The determination of the eigenfunctions have been less straightforward. In the first studies, numerically demanding image methods have been used, which limited the investigation to a small number of eigenmodes [2]. Even if simplifications have been done in the calculations, the methods presented in previous works do not allow to generate harmonics of arbitrary degree. This problem has been alleviated when computing only the spherical averages of the harmonics by the remark made in [7] and first proved in [8] that the number of modes of a given degree is sufficient in this case. It should be noted that the general construction of the eigenmodes has been given in [9], but failing to relate them to a basis more appropriate for the evaluation of the fluctuations of the microwave background. The two basis with their mutual relations were presented in [10], however with a level by level approach to the determination of the invariants.

In the following, I will first review how the peculiar nature of the four dimensional rotation group $SO(4)$, which is essentially the product of two copies of the group $SU(2)$, gives a basis for the functions on the three dimensional sphere, where the group Γ through which we factor has a simple expression. From the point of view of a given observer, an other basis with simple transformations through the rotations around this observer is more convenient. These two basis are related by the Clebsch–Gordan coefficients of $SU(2)$.

The functions on the quotients of the three-sphere are then described and arranged in irreducible representations of the finite subgroup Γ of the group $SU(2)$ through which we quotient. This allows to predict a simple form for the correlations of the microwave background in a suitable basis.

Finally, the building blocks for the invariants and the other irreducible representations of the finite subgroups of $SU(2)$ are given explicitly. This study extensively uses the description of the irreducible representations of $SU(2)$ by homogeneous polynomials in two variables. This allows for easy computations of the actions of $SU(2)$ by substitutions and the generation of new objects of interest by multiplication and other elementary operations on polynomials.

2 Functions on the three-sphere.

The three-sphere is remarkable among spheres because it can be given the structure of a non-Abelian group. This group can be considered either as the one of the quaternions of unit norm or the group of unitary two by two matrices $SU(2)$. As a group manifold, the three-sphere has symmetries stemming from the left and right actions on itself, which are different since the group is non-Abelian. The rotation group in four dimensions, the symmetry group of the three-sphere, is therefore factorized in a pair of factor homeomorphic to $SU(2)$, which will be denoted $SU(2)_L$ and $SU(2)_R$. Functions on the three-sphere form representations of this product group.

In order to make the description, it will be useful to parameterize the sphere with complex coordinates, stemming from the general form of matrices in the

group $SU(2)$:

$$p = \begin{pmatrix} s & -\bar{t} \\ t & \bar{s} \end{pmatrix} \tag{1}$$

The two complex numbers s and t verify the constraint $s\bar{s} + t\bar{t} = 1$. Unconstrained real coordinates for the 3-sphere can be taken as the phases of the complex variables s and t, ψ and ϕ, each varying in the interval $[0, 2\pi]$ and a third angle τ, varying in $[0, \pi/2]$ to denote their relative modules.

$$s = \cos\tau\, e^{i\psi}, \quad t = \sin\tau\, e^{i\phi}. \tag{2}$$

These angles are linked to the parameterization of three dimensional rotations by the Euler angles, which can be taken as $\psi + \phi$, 2τ and $\psi - \phi$. The metric in these coordinates take the simple form $dl^2 = d\tau^2 + \cos^2\tau d\psi^2 + \sin^2\tau\, d\phi^2$ and the volume element is $\sin\tau\cos\tau\, d\tau\, d\psi\, d\phi$.

From the matrix form (1), it appears that its four complex components form two doublets of the $SU(2)_L$ group, $(t, -s)$ and $(\bar{s}, \bar{t})$, and two of the $SU(2)_R$ group, $(s, -\bar{t})$ and $(t, \bar{s})$. They are in the tensor product of the fundamental representations of the two $SU(2)$ groups. When taking polynomials in the basic variables s and t and their complex conjugates, one obtains only product of identical representations of the left and right $SU(2)$ groups. One usually denotes them by a pair (j, j) of the spin with respect to each of the $SU(2)$ factors. Acting with the two commuting operators $t\partial_s - \bar{s}\partial_{\bar{t}}$ and $-\bar{t}\partial_s + \bar{s}\partial_t$ on s^{2j}, one obtains $(2j+1)^2$ harmonic polynomials of degree $2j$ which form a basis of the space of eigenvalue $2j(2j+2)$ of the Laplacian on the sphere. They can be characterized by the pairs (p, q) of the eigenvalues of the L_z operators in the two groups. Since j is a half integer, $n = 2j$ can take any integer value. In the variables τ, ψ, ϕ, the unnormalized eigenfunctions take a rather simple form:

$$\begin{aligned} Z_{npq}(\tau, \psi, \phi) &= (-1)^{\max(j+p, j-q)} R_{npq}(\tau) e^{i(q-p)\psi} e^{i(p+q)\phi} && (3) \\ R_{npq}(\tau) &= \cos^{|p-q|}\tau \sin^{|p+q|}\tau P^{|p-q|,|p+q|}_{\frac{1}{2}(n-|p+q|-|p-q|)}(\cos 2\tau) && (4) \end{aligned}$$

The $P^{\alpha,\beta}_r$ are polynomials of degree r, orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$, known as Jacobi polynomials. Since p and q are either both integers or both half-integers, Z is always 2π periodic in ϕ and ψ.

If the basis of eigenvectors of L^z_L and L^z_R is perfect for the intrinsic study of the harmonics, from the point of view of an observer, it is more convenient to make explicit the representations of the rotations around the observer. In the case of the three-sphere, these rotations form the diagonal subgroup of the full symmetry group, with the two $SU(2)$ factors identified. By the standard composition of the representations of $SU(2)$, the harmonics of degree n decompose into the representations with $l = 0$, 1,... up to n. Vectors are characterized by the pairs (l, m). The change of basis can be made explicit with the Clebsch–Gordan coefficients:

$$|n, l, m\rangle = \sum_{m_L, m_R} (jjm_L m_R|lm)|n, m_L, m_R\rangle \tag{5}$$

The coordinates in which those functions take simple forms are the (χ, θ, ϕ) variables, where χ is a measure of the distance to the observer and (θ, ϕ) the usual polar coordinates on the 2-sphere. In the change of variable from (τ, ψ, ϕ) to (χ, θ, ϕ), χ and θ only depend on τ and ψ and ϕ goes to $\phi - \pi/4$. The precise relation could be obtained from the following expression of the variables s and t in terms of (χ, θ, ϕ):

$$s = \cos\chi + i\sin\chi\cos\theta, \quad t = i\sin\chi\sin\theta\, e^{i\phi}. \tag{6}$$

The factor i, which gives a $\pi/4$ difference between the coordinates ϕ in the two systems of coordinates, allows to be compatible with the usual conventions. The χ dependences of the corresponding functions are expressed in terms of Gegenbauer polynomials of degree $n-l$ and do not vary with m. The Gegenbauer or ultraspherical polynomials are a special case of the Jacobi polynomials.

$$W_{nlm}(\chi, \theta, \phi) = i^l \sin^l \chi P_{n-l}^{l+\frac{1}{2}, l+\frac{1}{2}}(\cos\chi) Y_{lm}(\theta, \phi) \tag{7}$$

The factor i^l is necessary to be compatible with real Clebsch–Gordan coefficients according to the conventions of [11] that we adopt. The functions W_{nlm} are determined by their transformation property under the rotation group and are related to the functions Z_{npq} by eq. (5). The only difficulty is to verify the phase, but it can be determined by consideration of the leading terms of some special cases.

The linear contribution of a perturbation of the universe by such a mode to the cosmological background is simple: rays arriving at the observer are characterized by fixed θ and ϕ, so that the angular dependence is simply given by the Y_{lm} part. For the Sachs–Wolfe effect in the instantaneous approximation, the value of the χ dependent part at the value χ_{LS} corresponding to the surface of last scattering will appear. The details of the derivation and the computation of other contributions can be found elsewhere (see e.g., [7]), but what is important is that the time variation of the perturbations only depend on n and the radial one only on n and l: the modes which differ only by the value of m appear with the same weight.

3 Quotients of the three-sphere.

The definition of the Poincaré space and other cosmologically interesting quotients of the 3-sphere relies on the special structure of the four-dimensional rotation group as a direct product of two factors. Each of its $SU(2)$ factors acts without fixed points: any finite subgroup Γ of one of these factors also acts without fixed points so that the corresponding quotients of the three-sphere are regular manifolds. All the possible regular quotients of the three-sphere are not of this form, since double action manifolds appear in the classification done in the year 1932 in [12, 13]. Functions on the quotients are in one to one correspondence with functions on the three-sphere invariant under these finite subgroups.

Viewed as elements of the tensor product of representations of the two $SU(2)$ factors of $SO(4)$, the definition of invariants is simple: they are tensor product

of an invariant of the group Γ and an arbitrary vector. Modes of the Laplacian with eigenvalue $n(n+2)$ comes therefore in groups of $n+1$ since $n = 2j$.

The problem of the determination of the eigenmodes of the Laplacian for such a space therefore reduces to the determination of the invariants of the subgroup Γ in the representations of $SU(2)$.

However the knowledge of the invariants is not necessary if one is only interested in the spherically averaged fluctuations of the microwave background, their number in each representation of $SU(2)$ is sufficient. This stems from a sum rules for the coefficients of the eigenmodes first inferred from numerical evidence in [7] and proposed as a conjecture in their equation (25):

$$\sum_i \sum_{m=-l}^{l} \left|C^i_{nlm}\right|^2 = J(n)\frac{2l+1}{n+1} \tag{8}$$

Here, i indexes an orthonormal basis of the modes of the quotient of the sphere at level n and the C^i_{nlm} are the components of these modes in the basis of the W_{nlm} and $J(n)$ is the number of invariants of the group Γ. The right hand side is modified from the one given in [7] by the change to $n = \beta - 1$ and the total number of modes is $J(n)$ times $n+1$. A first proof was given in [8]. In this new proof, the symmetry of the Clebsh–Gordan coefficients expressed through the $3j$-symbols are used to obtain a sum rule for their squares.

We first remark that the relation (8) is simply verified in the case of the sphere. The group Γ being reduced to the identity, all vectors are invariant, $J(n)$ equals $n+1$ and the right hand side is simply $2l+1$. On the other hand, we can choose as a basis of the modes the one indexed by l and m, so that the sum on m in the left hand side is 1 or 0 according if i corresponds or not to the given l. Since there are $2l+1$ modes for each l, we get also $2l+1$ in the left hand side.

The fundamental identity which will be used is the following one:

$$\sum_{m_1,m_2} |(jj, m_0 m_1|l m_2)|^2 = \frac{2l+1}{2j+1} \tag{9}$$

If the sum were on m_0 and m_1, it would simply be the unitarity of the decomposition of the state $|l, m_2\rangle$ in the base of tensor products of two spins and the right hand side would be 1. The use of Wigner's $3j$-symbols allows to deduce eq. (9) from the unitarity condition. The $3j$-symbols are invariant by cyclic permutation of the three columns and take at most a sign for a permutation of two columns. They allow to express the Clebsch–Gordan coefficients as [11][1]

$$(j_1 j_2, m_1 m_2|lm) = (-1)^{j_1-j_2+m}\sqrt{2l+1}\begin{pmatrix} j_1 & j_2 & l \\ m_1 & m_2 & -m \end{pmatrix} \tag{10}$$

[1]Other choices of normalization are used. This one has the advantage of giving rise to real Clebsch–Gordan coefficients and uniform treatment of integer and half-integer values of l but implies that the eigenfunctions for l odd and m zero are imaginary. This is of no import in this work where we only consider absolute values, but a coherent set of phase choices must be made.

Hence we have:

$$(jj, m_0 m_1 | l m_2) = \pm \sqrt{\frac{2l+1}{2j+1}} (j m_0 | j l, -m_1 m_2) \tag{11}$$

Taking the square of this expression and summing on m_1 and m_2, we get the equation (9) using the unitarity of the decomposition of the vector $|jm_0\rangle$.

Suppose now that the invariant vector v of degree $2j$ of the group Γ is an eigenvector of L_z, $v = |m_0\rangle$, for a given m_0. The invariant functions form the family $|m_0, m_1\rangle$ with m_1 playing the role of the index i. The relation (8) is then simply the identity (9). A general invariant v is a linear combination of such vectors $|m_0\rangle$. The left hand side of the identity (9) is quadratic in the coordinates of v; the square terms give back the identity (9) for a normalized vector v while the interference terms vanish: terms with different m_0 contribute to C^i_{nlm} with differing values of m. Finally in the case where there are different invariants of the same degree j, the modes associated to each of the invariants are uncorrelated, so that their contributions simply add.

Using the relation (8) and the independence on m of the radial parts of the eigenmodes of the sphere, one can deduce that the contributions of eigenmodes of the Laplacian of degree $2j$ to the total power in degree l of the microwave background is the one of the corresponding modes on the sphere multiplied by the number of invariants of Γ of this degree divided by $2j+1$, up to the normalization of the initial power spectrum.

4 Correlations.

The power spectrum is however not sufficient to characterize the geometry. The effect of the geometry is important only for the lowest harmonics, which are plagued by their large cosmic variance and similar results can be obtained with quotients by the dodecahedral or cubic groups. One therefore want to compute the full correlation matrix of the spherical harmonics. For a microwave sky, this matrix is a really big object, since the total number of spherical harmonics grows quadratically with the maximum l considered, and the correlation matrix is quadratic in the number of objects considered. For the values of l from 2 to 10, we have 117 different spherical harmonics, which means 6903 coefficients in the full correlation matrix. The problem is double: how to compute efficiently all these numbers and how to compute the relevance of a given model for our present universe. We only have the 117 measured coefficients[2] of one realization to estimate the much bigger number of correlations. In the isotropic simply connected spaces, the problem simplifies a lot since the different harmonics are uncorrelated and the correlation matrix is fully determined by the 9 amplitudes of the $\langle C_l\rangle$, which can therefore be well constrained from the observations, even if the limited number of harmonics with small l limits the precision of the determination of the $\langle C_l\rangle$.

[2]In fact, the galactic foreground makes some of these measures impossible, roughly in the proportion of the solid angle that the galactic cut covers on the sky. This only worsen the problem.

In fact, in the quotients of the three-sphere, the homotopy group Γ appears also as a rotational symmetry around any observer. It is not a symmetry of the perturbations, but it is a symmetry of their correlations. This symmetry allows to characterize the correlation matrix with a much smaller number of coefficients, 310 in the octahedral case and 130 in the dodecahedral case. This simplification of the correlation matrix is made manifest when decomposing the spherical harmonics in irreducible representations of Γ.

At a given degree n, the eigenmodes of the space are a tensor product of an invariant v of the group Γ and a vector in the representation of dimension $n+1$ of $SU(2)$. We take as a basis of this representation of $SU(2)$ one adapted to its decomposition in irreducible representations of Γ. Its vectors $u_{r,i}$ are indexed by the representation r and the index i in the representation. We then have to decompose the tensor product $v \otimes u_{r,i}$ into the different representations of $SU(2)$ with l varying between 0 and n. As v is invariant under Γ, all the terms of this decomposition will transform under Γ as the vector of index i in a representation of type r. This mode will therefore contribute to fluctuations of the background which are themselves vector of index i in a component r of the decomposition of the spherical harmonics in representations of Γ.

With the usual assumption that the diverse modes of the universe are uncorrelated, we deduce that only vectors with the same index in the same representation of Γ can be correlated. Furthermore, the vectors with different indexes of a given representation have the same correlation matrix from the invariance under Γ of the whole procedure. The procedure for the explicit calculation of the correlations will be detailed in a further work.

If we retain the example of limiting ourselves to a maximum j of 10, we have in the case of the dodecahedron a total of two trivial representations, excluding the one of spin zero, five times the 3 representation, excluding the unobservable dipole at $l = 1$, ten times the 5 representation, eight times the 4_v and six times the $3'$ representations. The fact that the number of times each representation is present is roughly proportional to its dimension is exact asymptotically, as can be seen from the formulas of Kostant. The correlation among the fifty vectors appearing in 5 representations is given by five times the same symmetric ten by ten matrix, that is 55 coefficients. Adding the contribution for the other irreducible representations gives the announced number of 130 coefficients. The fact that this simplification of the correlation is manifest in a special basis is not a limitation in practice, since a change of basis is in any case necessary to rotate the coordinates from the galactic frame to the one adapted to the unknown orientation of the group Γ.

As we shall see in the following section, the determination of the basis elements can be reduced to the expansion of products of polynomials, an easy task for any computer algebra system.

5 Explicit formulas.

5.1 Generalities.

It is time to produce the explicit forms of the invariant vectors and the decomposition of the representations of $SU(2)$ in irreducible representations of the group Γ. The question of the invariants has been completely solved by Klein more than a century ago [3]. It was not however described in these terms, since Klein searched for invariant functions on the ordinary sphere under subgroups of the rotation group $SO(3)$. However his study is based on the description of the sphere as the complex projective space $\mathbb{P}^1(\mathbb{C})$ and the rotation of the sphere are represented as two by two matrices of complex numbers. These two by two matrices are to be taken projectively, but the unimodularity condition can be used to reduce this ambiguity to a sign. Klein remarked that in the case of the symmetry groups of regular polyhedrons, this choice of sign cannot be made in a way compatible with the group structure, so that one has to consider a double cover of the groups and he therefore studied finite subgroups of $SU(2)$, even if he did not use this modern terminology.

What is remarkable in Klein construction is that a unique invariant is sufficient to create the whole family of invariants. The irreducible representations of $SU(2)$ are symmetric tensor products of the fundamental one, and if we take two variables (s, t) that transform as a fundamental representation of $SU(2)$, all irreducible representations can be obtained as homogeneous polynomials in the variables s and t. Polynomials of total degree d give the $j = d/2$ representation of $SU(2)$. The generators of the Lie algebra of $SU(2)$ are written as:

$$L_z = \frac{1}{2}(s\partial_s - t\partial_t), \quad L_- = t\partial_s, \quad L_+ = s\partial_t \tag{12}$$

The monomial $s^p t^q$ has eigenvalue $(p-q)/2$ under L_z and its squared norm evaluate to $\binom{p+q}{p}^{-1} = p!q!/(p+q)!$. The action of the group $SU(2)$ is simply obtained by linear substitutions on the variables s and t. It is then evident that the product of invariant polynomials will give a new invariant polynomial. But other combinations are possible, based on the use of partial derivatives. The doublet (∂_s, ∂_t) transforms as a dual basis to the fundamental basis (s, t) and the properties of the invariant antisymmetric tensor ε of $SU(2)$ make $(-\partial_t, \partial_s)$ transform as (s, t). It then results that if the polynomials P and Q are both invariant under a subgroup of $SU(2)$, a new invariant polynomial is given by:

$$C(P, Q) = \begin{vmatrix} \partial_s P & \partial_s Q \\ \partial_t P & \partial_t Q \end{vmatrix} = \partial_s P \partial_t Q - \partial_t P \partial_s Q. \tag{13}$$

This expression being antisymmetric in P and Q, it cannot produce anything non trivial from a single invariant, but the Hessian can.

$$\mathrm{Hess}(P) = \begin{vmatrix} \partial_s^2 P & \partial_s\partial_t P \\ \partial_s\partial_t P & \partial_t^2 P \end{vmatrix} = \partial_s^2 P \partial_t^2 P - (\partial_s\partial_t P)^2. \tag{14}$$

From a first invariant of degree d, a second one with degree $2d - 4$ is obtained as the Hessian, and the combination of these two invariants by C gives a third

one of degree $3d-6$. This three invariants are sufficient to generate all invariant polynomials by multiplication. In fact, these equations are special cases of the formula of appendix A for the combination of two representations: equation (13) is the case $r=1$ and equation (14) the case $r=2$ with identical P and Q.

This method of Klein can be generalized to the problem of giving not only the invariants of Γ, that is the objects transforming in the trivial one-dimensional representation of Γ, but also all irreducible representations of Γ. Bertram Kostant made use of the McKay correspondence between finite subgroups of $SU(2)$ and Lie algebras of ADE type [14, 15, 16] to count the irreducible representations of the binary polyhedral groups in the representations of $SU(2)$ [4, 5, 17]. For our purpose it is however necessary to have explicit realizations of these representations, but the knowledge of the number of representations to find is an important guide. The structure of the generating functions for the multiplicities of a given representation is simple: it is a polynomial $P(t)$ divided by $(1-t^{2a})(1-t^{2b})$, with $2a$ and $2b$ the degrees of the first two invariant polynomials P and Q derived by Klein, which have no relation between them. If we take an irreducible representation of the group Γ realized as homogeneous polynomials of degree n, we can produce a new realization in degree $n+2ak+2bl$ by multiplying these polynomials by P^kQ^l. The formula derived by Kostant shows that this exhausts the occurrences of such a representation in any representation of $SU(2)$, if we have realizations of the representations of the degrees n appearing as exponents in the polynomial $P(t)$. Kostant proved that the number of such n is twice the dimension of the representation. In the following subsections we will give the explicit form of these representations with a sketch of the methods used to derive them.

The decomposition of any representation of $SU(2)$ in irreducible representations of Γ is therefore reduced to a combinatorial problem of multiplying polynomials to get objects of the desired degrees. The linear independence of the resulting realizations can be checked. The normalizations of the obtained polynomials cannot be predicted from the one of their components, and we therefore just try to have the simplest coefficients. As it should be, vectors belonging to distinct irreducible representations are automatically orthogonal. However, as soon as a given representation can be obtained in different ways at a given degree, we do not generally obtain orthogonal elements. In applications, it is generally simpler computationally not to build explicitly an orthonormal basis but to use the Gran–Schmidt matrix of scalar product to obtain basis independent results. Interestingly, these Gran–Schmidt matrices have rather peculiar arithmetic properties which we develop in the appendix B.

In the case of the cyclic group of order n and the dihedral group of order $4n$, the corresponding objects can be trivially written. The invariants for the cyclic group are st and s^n+t^n. All representations are of dimension 1 and appear two times, the trivial one with 1 and s^n-t^n, the $n-1$ others as s^i and t^{n-i}, with i varying between 1 and $n-1$. For the dihedral groups, the invariants are s^2t^2 and $s^{2n}+t^{2n}$. The trivial representation appears as 1 and $st(s^{2n}-t^{2n})$. A second one-dimensional representations appears with st and $s^{2n}-t^{2n}$. The two-dimensional representations have four realizations with i also varying between 1 and $n-1$: the doublets (s^i,t^i), $(s^{i+1}t,-st^{i+1})$, $(t^{2n-i},(-1)^is^{2n-i})$

Table 1: Representations of the binary tetrahedral group.

1	(1) $(t^{12} - 33\, s^4\, t^8 - 33\, s^8\, t^4 + s^{12})$
2	(s, t) $(s^5 - 5\, s\, t^4, t^5 - 5\, s^4\, t)$ $(-7\, s^4\, t^3 - t^7, s^7 + 7\, s^3\, t^4)$ $(11\, s^8\, t^3 + 22\, s^4\, t^7 - t^{11}, s^{11} - 22\, s^7\, t^4 - 11\, s^3\, t^8)$
3	$(s^2, s\, t, t^2)$ $(4\, s\, t^3, s^4 - t^4, -4\, s^3\, t)$ $(s^6 - 5\, s^2\, t^4, -2(s^5\, t + s\, t^5), t^6 - 5\, s^4\, t^2)$ $(t^6 + 3\, s^4\, t^2, -4\, s^3\, t^3, s^6 + 3\, s^2\, t^4)$ $(-2\, s\, t^7 - 14\, s^5\, t^3, s^8 - t^8, 2\, s^7\, t + 14\, s^3\, t^5)$ $(t^{10} - 14\, s^4\, t^6 - 3\, s^8\, t^2, 8\, s^3\, t^7 + 8\, s^7\, t^3, -3\, s^2\, t^8 - 14\, s^6\, t^4 + s^{10})$
$2_{1,2}$	$(-t^3 \mp i\, \sqrt{3}\, s^2\, t, s^3 \pm i\, \sqrt{3}\, s\, t^2)$ $(s^5 + s\, t^4 \pm 2\, i\, \sqrt{3}\, s^3\, t^2, t^5 + s^4\, t \pm 2\, i\, \sqrt{3}\, s^2\, t^3)$ $(-t^7 + 5\, s^4\, t^3 \pm i\, \sqrt{3}\, (s^6\, t + 3\, s^2\, t^5), s^7 - 5\, s^3\, t^4 \mp i\, \sqrt{3}\, (s\, t^6 + 3\, s^5\, t^2))$ $\left(s\, t^8 - 10\, s^5\, t^4 + s^9 \mp 4\, i\, \sqrt{3}\, (s^3\, t^6 + s^7\, t^2), t^9 - 10\, s^4\, t^5 + s^8\, t \mp 4\, i\, \sqrt{3}\, (s^2\, t^7 + s^6\, t^3)\right)$
$1_{1,2}$	$(s^4 + t^4 \pm 2\, i\, \sqrt{3}\, s^2\, t^2)$ $(t^8 - 10\, s^4\, t^4 + s^8 \mp 4\, i\, \sqrt{3}(s^6\, t^2 + s^2\, t^6))$

and $(st^{2n-i+1}, (-1)^{i+1} s^{2n-i+1} t)$. The last two representations depend on the parity of n, since they are bosonic for n even and fermionic otherwise. For n even, we have one representation given by $s^n + t^n$ or $st(s^n - t^n)$ and the other given by $s^n - t^n$ or $st(s^n + t^n)$. For n odd, the imaginary unit i appears. We therefore have $s^n + it^n$ or $st(s^n - it^n)$ for one representation and $s^n - it^n$ or $st(s^n + it^n)$ for the other.

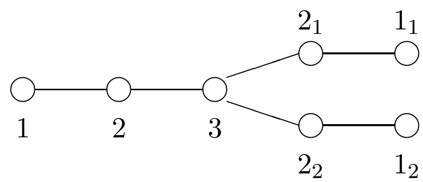
5.2 Tetrahedron and octahedron.

The tetrahedral group is generated by i, j and $1/2(1 + i + j + k)$. In degree 4, $s^4 + t^4$ and $s^2 t^2$ are invariant under i and j and we can form with them covariants of the group, that is objects transforming under the non-trivial one-dimensional representations of the tetrahedral group $T = s^4 + t^4 \pm 2i\sqrt{3}s^2 t^2$. From them every representations can be formed.

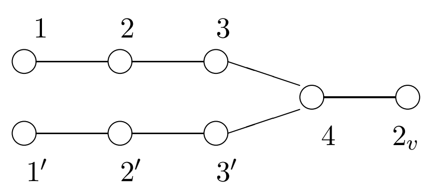
The squares give the two other required instances of the one-dimensional representations. Acting on them with the partial derivatives $-\partial/\partial t$ and $\partial/\partial s$, or multiplying them with s and t yields the four instances of the 2_1 and 2_2 representations. Their product gives the invariant of degree 8, $P = s^8 + 14s^4 t^4 + t^8$ and their combination with the cross product C gives the invariant of degree 6, $Q = st^5 - s^5 t$. The cross product of these two invariants gives the invariant of degree 12, $s^{12} - 33s^8 t^4 - 33s^4 t^8 + t^{12}$, which can also be obtained from the

real part of the third power of T, the imaginary part being proportional to Q^2. The fundamental representation is obtained by acting with the derivatives on the three invariants of degree 6, 8 and 12. The vector representation can be obtained from the cross product of the basic one (s^2, st, t^2) with the two invariants P and Q, and also by applying the triplet $(\partial^2/\partial s^2, -\partial^2/\partial s\partial t, \partial^2/\partial s^2)$ on the three invariants.

It is therefore possible to construct the full decomposition of the representations of $SU(2)$ under the tetrahedral group without having to make explicit calculations of the transformations, apart from the covariance property of T. The necessary polynomials are collected in table (1). The labeling of the representations is given on the extended E6 diagram, following the McKay correspondence.



The corresponding decomposition for the octahedral group is rather simple now, since the tetrahedral group is a subgroup of index two. The representations 1_1 and 1_2 of the tetrahedral group merge in a single two dimensional representation 2_V, the representations 2_1 and 2_2 give the four dimensional representation. The 1, 2 and 3 representations of the tetrahedral group each appear in two different versions as representations of the octahedral group, which can be interchanged by any object in the $1'$ representation. The relation between representations of the binary tetrahedral group and the binary octahedral group appear in the following unusual disposition of the extended E7 Dynkin diagram.



A simple multiplication by the covariant Q, which transform in the $1'$ representation of the octahedral group allows to obtain all necessary construct for the decomposition under this group. We will not write them explicitly.

5.3 Dodecahedral group

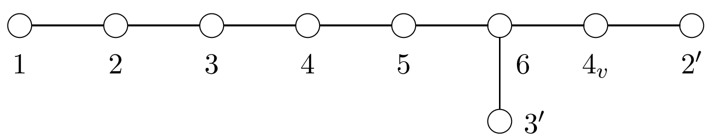
The corresponding calculations for the dodecahedral group are more complex, since the representations are bigger and have more different avatars. In order to limit the place taken, I will only give explicitly the vectorial representations, which are the only necessary ones for our project, apart from the alternative two dimensional one, which is useful to build the vectorial four dimensional representation.

As before we need to make a specific choice for the orientation of the polyhedron. It will be very convenient to follow Klein in choosing to orient a five

fold symmetry axis along the z-axis. The orientation is completely fixed by choosing to make a two fold symmetry axis coincide with the y-axis. This single out a ten element dihedral subgroup of the dodecahedral group or rather its twenty elements double cover as a subgroup of the binary dodecahedral group. One particularly useful property of this orientation is that, apart in one case, the eigenvector spaces of the order five rotation are one-dimensional in the irreducible representations of the dodecahedral group, the exception being the one corresponding to the eigenvalue -1 in the six-dimensional spinorial representation. This allows for an easy link between the vectors of different realizations of the same irreducible representation. The full group can then be generated from this dihedral subgroup and a single supplementary element, since no subgroup of the dodecahedral group has index 2 or 3. Our choice is for a two fold symmetry with an axis in the xz plane. Apart from the normalization factor, which is never very important, it can be chosen with rather simple coordinates:

$$K' = \frac{1+\sqrt{5}}{2} k - i \tag{15}$$

According to the McKay correspondence, the irreducible representations of the dodecahedral group label the vertex of the extended E8 diagram:



The first step is to determine the lowest dimensional non trivial splitting, which occurs for the homogeneous polynomials of degree 6, the representation $l = 3$ of $SU(2)$. This is made easy because s^3t^3 belongs to the $3'$ representation and the doublet (s^4t^2, s^2t^4) belongs to the vectorial 4_v representation. Use of the transformation in (15) then allows to complete the representations. The $3'$ has a basis given by

$$(3\,s\,t^5 + s^6, 5\,s^3\,t^3, t^6 - 3\,s^5\,t). \tag{16}$$

The basis of the 4_v representation is

$$(-t^6 - 2\,s^5\,t, 5\,s^4\,t^2, 5\,s^2\,t^4, s^6 - 2\,s\,t^5). \tag{17}$$

The factors of 5 are chosen to simplify the transformations and to have simple relations between the norms of the basic vectors. As it should be, vectors belonging to different representations are orthogonal. The fact that the 4_v representation is the tensor product of the basic one and the second two dimensional one $2'$ allows to determine the basis of the $2'$ representation with polynomials of degree 7. They should reproduce the basis elements of the degree 6 representation 4_v upon differentiation with respect to s and t. One obtains

$$(t^7 + 7\,s^5\,t^2, s^7 - 7\,s^2\,t^5) \tag{18}$$

A representation 4_v in degree 8 is obtained by multiplying this vector by s and t. From the $3'$ representation, we can form an invariant norm, which is the

Table 2: Representations of the binary dodecahedral group.

1	(1) $(t^{30} - 522\, s^5\, t^{25} - 10005\, s^{10}\, t^{20} - 10005\, s^{20}\, t^{10} + 522\, s^{25}\, t^5 + s^{30})$
3	$(s^2, s\, t, t^2)$ $(30\, s^6\, t^4 - 10\, s\, t^9, t^{10} - 36\, s^5\, t^5 - s^{10}, 30\, s^4\, t^6 + 10\, s^9\, t)$ $(-11\, s^2\, t^{10} + 66\, s^7\, t^5 + s^{12}, -5\, s\, t^{11} - 5\, s^{11}\, t, t^{12} - 66\, s^5\, t^7 - 11\, s^{10}\, t^2)$ $(t^{18} + 126\, s^5\, t^{13} + 117\, s^{10}\, t^8 - 12\, s^{15}\, t^3, -45\, s^4\, t^{14} - 130\, s^9\, t^9 + 45\, s^{14}\, t^4, 12\, s^3\, t^{15} + 117\, s^8\, t^{10} - 126\, s^{13}\, t^5 + s^{18})$ $(2\, s\, t^{19} + 342\, s^6\, t^{14} + 494\, s^{11}\, t^9 - 114\, s^{16}\, t^4, t^{20} + 114\, s^5\, t^{15} + 114\, s^{15}\, t^5 - s^{20}, -114\, s^4\, t^{16} - 494\, s^9\, t^{11} + 342\, s^{14}\, t^6 - 2\, s^{19}\, t)$ $(t^{28} - 360\, s^5\, t^{23} - 4370\, s^{10}\, t^{18} - 1035\, s^{20}\, t^8 + 12\, s^{25}\, t^3, 75\, s^4\, t^{24} + 2300\, s^9\, t^{19} + 2300\, s^{19}\, t^9 - 75\, s^{24}\, t^4, \ldots)$
5	$(s^4, 2\, s^3\, t, 3\, s^2\, t^2, 2\, s\, t^3, t^4)$ $(2\, s^6\, t^2 - 4\, s\, t^7, t^8 - 8\, s^5\, t^3, 15\, s^4\, t^4, -8\, s^3\, t^5 - s^8, 2\, s^2\, t^6 + 4\, s^7\, t)$ $(6\, s^2\, t^8 - 8\, s^7\, t^3, 2\, s\, t^9 + 14\, s^6\, t^4, -t^{10} - s^{10}, 2\, s^9\, t - 14\, s^4\, t^6, 8\, s^3\, t^7 + 6\, s^8\, t^2)$ $(10\, s^3\, t^3\, (t^6 - 3\, s^5\, t), (3\, s\, t^5 + s^6)^2, 3\, s\, t^{11} + 42\, s^6\, t^6 - 3\, s^{11}\, t, -(t^6 - 3\, s^5\, t)^2, -10\, s^3\, t^3\, (3\, s\, t^5 + s^6))$ $(-11\, s^4\, t^{10} + 66\, s^9\, t^5 + s^{14}, -16\, s^3\, t^{11} + 66\, s^8\, t^6 - 4\, s^{13}\, t, -15\, s^2\, t^{12} - 15\, s^{12}\, t^2, -4\, s\, t^{13} - 66\, s^6\, t^8 - 16\, s^{11}\, t^3, t^{14} - 66\, s^5\, t^9 - 11\, s^{10}\, t^4)$ $(17\, t^{16} + 1092\, s^5\, t^{11} + 364\, s^{10}\, t^6 - 4\, s^{15}\, t, -910\, s^4\, t^{12} - 1040\, s^9\, t^7 + 60\, s^{14}\, t^2, 420\, s^3\, t^{13} + 1755\, s^8\, t^8 - 420\, s^{13}\, t^3, \ldots)$ $(-2\, s\, t^{17} - 182\, s^6\, t^{12} - 104\, s^{11}\, t^7 + 4\, s^{16}\, t^2, -t^{18} + 14\, s^5\, t^{13} + 143\, s^{10}\, t^8 - 28\, s^{15}\, t^3, 75\, s^4\, t^{14} + 75\, s^{14}\, t^4, \ldots)$ $((30\, s^6\, t^4 - 10\, s\, t^9)^2, 2\, (30\, s^6\, t^4 - 10\, s\, t^9)\, (t^{10} - 36\, s^5\, t^5 - s^{10}), 2\, t^{20} - 444\, s^5\, t^{15} + 3388\, s^{10}\, t^{10} + 444\, s^{15}\, t^5 + 2\, s^{20}, \ldots)$ $(-2\, s^3\, t^{19} - 342\, s^8\, t^{14} - 494\, s^{13}\, t^9 + 114\, s^{18}\, t^4, -3\, s^2\, t^{20} - 456\, s^7\, t^{15} - 494\, s^{12}\, t^{10} + s^{22}, -3\, s\, t^{21} - 342\, s^6\, t^{16} - 342\, s^{16}\, t^6 + 3\, s^{21}\, t, \ldots)$ $(21\, t^{26} - 5060\, s^5\, t^{21} - 37145\, s^{10}\, t^{16} - 1610\, s^{20}\, t^6 + 2\, s^{25}\, t, 2300\, s^4\, t^{22} + 43700\, s^9\, t^{17} + 9200\, s^{19}\, t^7 - 50\, s^{24}\, t^2, -600\, s^3\, t^{23} - 32775\, s^8\, t^{18} - 32775\, s^{18}\, t^8 + 600\, s^{23}\, t^3, \ldots)$
$3'$	$(3\, s\, t^5 + s^6, 5\, s^3\, t^3, t^6 - 3\, s^5\, t)$ $(10\, s^8\, t^2 - 20\, s^3\, t^7, t^{10} + 14\, s^5\, t^5 - s^{10}, 10\, s^2\, t^8 + 20\, s^7\, t^3)$ $(t^{14} + 14\, s^5\, t^9 + 49\, s^{10}\, t^4, -7\, s^2\, t^{12} - 48\, s^7\, t^7 + 7\, s^{12}\, t^2, 49\, s^4\, t^{10} - 14\, s^9\, t^5 + s^{14})$ $(3\, s\, t^{15} + 143\, s^6\, t^{10} - 39\, s^{11}\, t^5 - s^{16}, 25\, s^3\, t^{13} + 25\, s^{13}\, t^3, -t^{16} + 39\, s^5\, t^{11} + 143\, s^{10}\, t^6 - 3\, s^{15}\, t)$ $(-52\, s^3\, t^{17} - 442\, s^8\, t^{12} - 544\, s^{13}\, t^7 + 14\, s^{18}\, t^2, -t^{20} + 136\, s^5\, t^{15} + 136\, s^{15}\, t^5 + s^{20}, 14\, s^2\, t^{18} + 544\, s^7\, t^{13} - 442\, s^{12}\, t^8 + 52\, s^{17}\, t^3)$ $(-t^{24} + 112\, s^5\, t^{19} + 646\, s^{10}\, t^{14} - 1292\, s^{15}\, t^9 + 119\, s^{20}\, t^4, 5\, s^{22}\, t^2 - 5\, s^2\, t^{22} - 570\, (s^7\, t^{17} + s^{17}\, t^7), \ldots)$
$2'$	$(t^7 + 7\, s^5\, t^2, s^7 - 7\, s^2\, t^5)$ $(-26\, s^3\, t^{10} - 39\, s^8\, t^5 + s^{13}, -t^{13} - 39\, s^5\, t^8 + 26\, s^{10}\, t^3)$ $(-t^{17} + 119\, s^5\, t^{12} - 187\, s^{10}\, t^7 + 17\, s^{15}\, t^2, -17\, s^2\, t^{15} - 187\, s^7\, t^{10} - 119\, s^{12}\, t^5 - s^{17})$ $(-46\, s^3\, t^{20} - 1173\, s^8\, t^{15} + 391\, s^{13}\, t^{10} - 207\, s^{18}\, t^5 - s^{23}, t^{23} - 207\, s^5\, t^{18} - 391\, s^{10}\, t^{13} - 1173\, s^{15}\, t^8 + 46\, s^{20}\, t^3)$

invariant polynomial of degree 12, $P = s\,t^{11} - 11\,s^6\,t^6 - s^{11}\,t$. The Hessian gives then the second invariant of degree 20,

$$t^{20} + 228\,s^5\,t^{15} + 494\,s^{10}\,t^{10} - 228\,s^{15}\,t^5 + s^{20}$$

The three antisymmetric combinations of the basis functions of the first $3'$ representations by the operation of (13) form a new representations $3'$ in degree 10. A third version of the $3'$ representation appears as the symmetric square of the $2'$ representation, the one of degree 14. The three other versions of the $3'$ representation can be obtained by combination with the invariant P, adding 10 to the degree. The $2'$ representation in degree 13 can be obtained by combining the $3'$ representation in degree 6 and the $2'$ representation in degree 7, but it is simpler to use the combination method of the appendix A. This is also handy to produce all the occurrences of the 3 and 5 representations from their fundamental case and the invariants. Finally, the useful representations are summarized in table 2. For lack of space, some of the representations are not complete. The other basis elements can be obtained from the written ones by using the rotation around the y axis, which stems from the simple substitution $s \to t$ and $t \to -s$. The 4_v and the 6 representations can be obtained from the $2'$ and $3'$ representations, by using the two possible ways of tensoring by the fundamental representation, that is multiplying each element either by s or t, or differentiating by $-\partial_t$ and ∂_s.

A method for the decomposition of the representation of $SU(2)$ in irreducible representations of the binary icosahedral group was proposed in [18]. With respect to the present work, it presents however strong limitations, since it is based on diagonalizing an operator. Explicit solutions become difficult to obtain as soon as the blocks to diagonalize have a rank bigger than 2 or 3. Furthermore, since irreducible representations are recognized as the eigenspaces of the operator, they do not come with a standard basis in which the representation of the group Γ is fixed. This does not allow to use the symmetry under Γ to relate the correlations for different vectors of the same representation.

6 Conclusion

An explicit decomposition of the representations of the group $SU(2)$ in irreducible representations of its finite subgroups has been provided. This allows not only to produce all eigenmodes of the quotients of the three-sphere, but also to group them in irreducible representations of the group Γ. In this way, the determination of the correlation matrix of the different modes of the CMB becomes accessible. The kind of analysis which has only be made for toroidal geometries, allowing for a maximal use of the available evidence to differentiate the possible geometries, can now be done also for spherical geometries.

Acknowledgments: It is a pleasure to thank Olivier Babelon and Jean-Bernard Zuber who gave me access to the references [3, 4]. The exigence of an anonymous referee contributed to the clarity of this work.

A Representation of $SU(2)$ as homogeneous polynomials.

It is a well known fact that the irreducible representations of $SU(2)$ can all be obtained as symmetrical tensor products of the fundamental one. This has motivated the expression of the components of any representation as multispinors and has been extensively used to deduce properties of the representation matrix and the Clebsch–Gordan coefficients for the decomposition of tensor products of representations, see e.g., [11]. But polynomials are just that, the symmetric product of objects which can be considered as basis elements in a fundamental representation. They are very interesting for automated treatments since most computer algebra systems have in their kernel of functions the operations on polynomials.

The action of the group $SU(2)$ on any object is obtained through substitution for the basic variables of their linear transform. This is quite easy. What is not so straightforward is the expression of the decomposition of tensor product of representations in their irreducible factors. Let us consider homogeneous polynomials $P(s,t)$ of degree p and $Q(s,t)$ of degree $q \leq p$, representing the objects of spin $p/2$ and $q/2$ for which we wish the decomposition in irreducible components. The highest spin component is easily obtained, since the product PQ gives a polynomial with degree $p+q$. The lowest spin component can also be inferred in substituting in Q the derivatives $-\partial_t$ and ∂_s to the variables s and t: $Q(-\partial_t, \partial s)P(s,t)$ is a polynomial of degree $p-q$.

In the general case, the idea is to enlarge the representation by a factor which will be exactly compensated by a similar one in the other one. This role is played by differential operators of order r acting on P and Q combined to form an invariant of $SU(2)$. The representation of degree $p+q-2r$ is therefore obtained through

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k} \frac{\partial^r}{\partial s^k \partial t^{r-k}} P(s,t) \frac{\partial^r}{\partial s^{r-k} \partial t^k} Q(s,t) \tag{19}$$

Apart from a normalization factor which depends on p, q and r, this formula can be shown to be equivalent to the usual formulas for $3j$-symbols. This normalization could be obtained recursively, taking for P the monomial s^p and for Q the monomials $s^{q-l}t^l$ for successive values of l, but I shall not need it in this work. The primary representations which we obtain do not need normalization and only relative normalization will be used in the study of the asymmetries of the microwave background. The use of the non-normalized basis of the homogeneous polynomials has the advantage of ridding ourselves of all square roots factors. This is computationally interesting since the extraction of the squared factors under square roots is much more demanding that purely rational calculations.

B Remarks on the norms.

It is quite remarkable that the squared norm of the objects created in section (5) are peculiar rational numbers. The numerator is, apart from small eventual powers of 2 and 3, a power of 5 and the denominator is the product of primes smaller than the degree of the object. This pattern seems to fail for bigger degrees, but it can be observed that the failure happens when there are independent objects of the same degrees with the same transformation properties. In this case there is an arbitrariness in the choice of the basis functions, so that it is natural that the objects obtained by arbitrary choices do not possess any particular property.

An intrinsic description of the space of the invariants can be obtained with a projector. It does not depend on any base choice and the matrix elements of the projector are then rational numbers with denominators which are essentially powers of 5, i.e., eventually multiplied by 3, 2 or 4. In the case of the tetrahedron group or the octahedron group, the same pattern appears with 5 replaced by 2.

This suppose that we express the projector in the basis of the homogeneous polynomials, which is not normalized. Otherwise we obtain a symmetrized matrix with square roots appearing in the non-diagonal elements.

This pattern can be understood since the projector on the space of invariants can be obtained as the mean of the group elements in the group algebra. Then we can use the explicit expression given by Klein for the elements of the group which are not in the binary dihedral group of order 20, in term of a fifth root of the unity ϵ.

$$\sqrt{5}\begin{pmatrix} s' \\ t' \end{pmatrix} = \pm \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix}^{\nu} \begin{pmatrix} -\epsilon+\epsilon^4 & \epsilon^2-\epsilon^3 \\ \epsilon^2-\epsilon^3 & \epsilon-\epsilon^4 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix}^{\mu} \begin{pmatrix} s \\ t \end{pmatrix} \tag{20}$$

$$\sqrt{5}\begin{pmatrix} s' \\ t' \end{pmatrix} = \pm \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix}^{\nu} \begin{pmatrix} -\epsilon^2+\epsilon^3 & -\epsilon+\epsilon^4 \\ -\epsilon+\epsilon^4 & \epsilon^2-\epsilon^3 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix}^{\mu} \begin{pmatrix} s \\ t \end{pmatrix} \tag{21}$$

The matrix notations are not those of Klein and the exponents μ and ν take 5 different values. When we take for ϵ a different fifth root of the unity, the elements described by eqs. (20,21) are permuted among themselves. The sum on the elements of the group is therefore independent on the choice of the root ϵ, that is an invariant of the Galois group and therefore a rational number. Furthermore, ϵ is an integral element of its field, since it is a solution of an equation with integer coefficients with leading coefficient 1. When we apply the transformations (20,21) to homogeneous polynomials of degree $2n$, the $\sqrt{5}$ factor gives a 5^n denominator and the numerator is polynomial in ϵ, s and t with integer coefficients, so that when we sum over the group elements we obtain a polynomial in s and t with integer coefficients for the numerator. In the basis of the monomials in s and t, the projection on the invariants is therefore given by a matrix with integer coefficients divided by 60.5^n. The factor of 60 stems from the division by the order of the group.

In the case of non-trivial representations of the group Γ, the projections on specific basis elements of the representation as well as the elementary matrix sending from one basis element to an other should be expressible as elements of the group algebra and a similar result should be proved. However the coefficients

of these elements of the group algebra are no longer rational and the invariance of the resulting matrices under the Galois group depends on a subtle relation between the exchange of coefficients and of group elements under the Galois group. The cases of the tetrahedron or octahedron groups are similar.

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